Multiplicity of 1 in Laplacian Spectra of trees

Abstract

In this paper, we interpret the multiplicity of 1 in Laplacian spectra of trees and prove that Faria's inequality turns to an equality in the case of normal trees which yields that in any tree without a vertex of degree 2, Faria equality holds and multiplicity of 1 in Laplacian spectrum will be equal to star degree of the the tree. As a result we introduce a combinatorial procedure for computing the multiplicity of 1 . In the way to prove this results we will introduce many transformation on graphs which are invariant regarding to the multiplicity of 1. We also introduce an inequality for the multiplicity of 0 in adjacency spectrum of graphs and again introduce a procedure to compute this multiplicity in the case of trees.

Keywords: Laplacian spectrum, Tree, Eigenvalue Multiplicity.

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1 Introduction

Throughout this paper, G denotes a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is the number of vertices of G. the *open neighborhood* of v is the set $N[v] = \{u \in V(G) \,|\, uv \in E(G)\}$. Let G_1 and G_2 be two disjoint graphs and $v \in V(G_1)$ and $u \in V(G_2)$, then by $G_1v : uG_2$ we mean a new graph which has the vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{u, v\}$. Let G be a graph of order n and $A(G)$ be the $(0, 1)$-adjacency matrix of G . The matrix $Q(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G, where $D(G)$ is the $n \times n$ diagonal matrix with $\{d(v_1), \ldots, d(v_n)\}$ as diagonal entries (and all other entries 0). The *characteristic polynomial* $P(G, \lambda)$ of G is the polynomial $det(\lambda I - A(G))$, where I is the identity matrix of size n. Since $A(G)$ and $Q(G)$ are real and symmetric, their eigenvalues are real and called the *adjacency eigenvalues* and the *Laplacian eigenvalues* of G, respectively. Assume that $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_n = 0$ are the adjacency and the Laplacian eigenvalues of G, respectively. The multiset of eigenvalues of $A(G)(Q(G))$ is called the *adjacency* (*Laplacian*) *spectrum*

of G. Two graphs are said to be *cospectral* with respect to adjacency (Laplacian) matrix if they have the same adjacency (Laplacian) spectrum. A graph is said to be *determined by the adjacency* (*Laplacian*) *spectrum* if there is no other non-isomorphic graph with the same adjacency (Laplacian) spectrum. We call a graph G *normal*, if no two vertices of degree two are adjacent in G.

The concept of eiganvalue multiplicity has been extensively studied by several authors. The multiplicity of 0 in adjacency spectrum is important for trees as it determines the maximum matching size of trees.[theorem8.1, cev-doob-sach]. In [3]it is proved that:

$$p(T) - 1 \geq \tilde{m}_T(0) \geq p(T) - q(T) \tag{1}$$

Where T is a tree and $\tilde{m}_G(\lambda)$ denotes the multplicity λ in adjacency spectrum of G. $p(G)$ is the number of pendant vertices of G and $q(G)$ is the number of quasi-pendant vertices of G(the vertices with a leaf neighbour). There is a similiar result for multplicity of 1 in Laplacian spectrum of a graph. In [6] Faria observed that for any graph G:

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$$m_G(1) \geq p(G) - q(G)$$

Where she also mentioned that "the multiplicity of the root 1 of $det(xI-B)$ or $det(xI-L)$ is greater than or equal to p. However, we cannot guarantee that the multiplicity is exactly p." In his context, $m_G(\lambda)$ denotes the multplicity λ in Laplacian spectrum of G, B and L are signless Laplacian matrix and Laplacian matrix respectively. In this paper $p(G) - q(G)$ refered as the "star degree" of the graph. This result explains why 1 is often an eigenvalue, and so often a multiple eigenvalue of trees. In [4]it is stated that "it is worth recalling that as trees are not regular we do not expect a direct relationship between the two spectra. The similarity between these two inequalities is a little unusual." In [8], Grone, Merris, and Sunder proved that for a tree T with n vertices, if $\lambda > 1$ is an integer Laplacian eigenvalue of T , then $\lambda \mid n$ and $m_T(\lambda) = 1$. In that paper, it is also pointed out that "there is an abundance of examples that leads the authors to believe there can be no simple graph theoretic interpretation for $m_T(1)$. In [8] the following result obtained for trees,

$$m_T(1) \leq p(T) - 1$$

Making Faria's inequality a double-side inequality. Recently, Shao, Guo and Shan [9] investigated the effect on the multiplicity of Laplacian eigenvalues of connected graphs when adding edges and Gou et al. [10]investigated the effect of adding one edge between

two disjoint graphs and concluded some results about special subgraphs of a tree that could be removed with a moderated change in the multiplicity of the remaining graph. In this paper we show that for each normal tree T we have the following:

$$m_T(1) = p(T) - q(T)$$

and as a result we will determine the multiplicity of 1 as a Laplacian eigenvalue in all trees.

In this paper we introduce a procedure on trees to compute the multiplicity of 0 in adjacency spectrum and multiplicity of 1 in Laplacian spectrum. We further the result by Nosal [3] and prove the inequality 1 for all graphs:

$$p(G) - 1 \geq \tilde{m}_G(0) \geq p(G) - q(G)$$

We also provide a sufficient condition for the trees which has the star-degree equal to multiplicity of 1 in Laplacian spectrum.

2 Preliminaries

We will introduce some new modifications on graphs which preserve the multiplicity of 1. Though some results in this section are proved in previous works[8, 10] we will introduce a shorter proof for these results.

Lemma 1. *Let G be a graph. For any vertex v and any Laplacian eigenvalue μ and the corresponding eigenvector x, we have:*

$$(d_v - \mu)x_v = \sum_{u \sim v} x_u \tag{2}$$

were x_v and d_v denote the corresponding value of x and degree for vertex v respectively.

Proof. Simple matrix multiplication. □

Corollary 1. *If v is a pendant vertex of G and u is its neighbour in G, and $\vec{x}$ is a Laplacian vector corresponding to eigenvalue $\mu = 1$, then $x_u = 0$*

Remark 1. By corollary 1 one can deduce that the only factor which determines the element of eigenvector corresponding to a vertex is the corresponding elements of eigenvecor of its neighbours.

Lemma 2. *Let G_1 and G_2 be two vertex-disjoint graphs and let $G = G_1us : tvG_2$ be the graph obtained by joining the vertex u of the graph G_1 to the vertex s of a new path P_2 and connecting the other end of P_2, t to vertex v of the graph G_2 by an edge and let G' be the graph obtained by contracting the edge $\{u, v\}$ in $G_1u : vG_2$. Then:*

$$m_G(1) = m_{G'}(1)$$

Proof. Suppose $\vec{x}$ is a Laplacian eigenvector for $\mu = 1$ for G. According to equation 2 We have the following equations for G:

$$\begin{cases} x_s = x_u + x_t \\ x_t = x_v + x_s \end{cases} \Longrightarrow x_u = -x_v$$

Suppose $N_{G_1}(u) = \{z_1, z_2 ..., z_{d_u}\}$ and $N_{G_2}(v) = \{w_1, w_2 ..., w_{d_v}\}$. Then again by (2) we have:

$$\begin{cases} (d_u - 1)x_u = \sum\limits_i x_{z_i} + x_s \\ -(d_v - 1)x_v = -\sum\limits_i x_{w_i} - x_t \end{cases} \Rightarrow (d_u + d_v - 3)x_u = \sum_i x_{z_i} - \sum_i x_{w_i}$$

In the case that u or v don't have any neighbours we consider $\sum\limits_i x_{z_i} = 0$ and $\sum\limits_i x_{w_i} = 0$ respectively. Now define $\vec{x'}$ for G' as follows:

$$x'_i = \begin{cases} x_i & i \in G_1 \backslash \{u\} \\ -x_i & i \in G_2 \backslash \{v\} \\ x_u & i = u = v \end{cases}$$

It is easy to check that $\vec{x'}$ is an eigenvector for G'. The procedure above is reversible. In fact we have an isomorphism from the eigenspace of 1 in G to eigenspace of 1 in G' with the above translation. Therefore the result follows.

□

Corollary 2. *[8] If G is a graph obtained from G' by connecting one of the endvertices of P_3 to an arbitrary vertex of G, then we have:*

$$m_G(1) = m_{G'}(1)$$

Corollary 3. *For every positive integer* $m_{P_n}(1) = 1$ *if and only if* $3|n$. *Otherwise* $m_{P_n}(1) = 0$.

Proof. Use previous lemma with an induction on n. □

Lemma 3. *Let* G *be a graph,* u *a quasi-pendant vertex of* G *and* $v \in V(G)$ *be a pendant vertex adjacent to* u. *Also suppose that* $G \backslash \{u, v\}$ *has* k *components which are* $G_1, G_2, ..., G_k$. *Now construct* $G'_1, G'_2, ..., G'_k$ *by adding* u_i *and* v_i *to them as* u *and* v *were connected in* G *and define* $\widetilde{G} = \bigcup_{i=1}^{k} G'_i$, *Then:*

$$m_G(1) = m_{\widetilde{G}}(1)$$

Proof. Suppose $\vec{x'_j}$ is a Laplacian eigenvector of $\mu = 1$ for G'_j. As G'_js are independent all of $\vec{x'_j}$ we could build a new eigenvector for $\widetilde{G}$ by concatenating eignevectors for G'_js. Construct $\vec{x}$ for G with elements as follows: By x'_{i_j} we mean the element of $\vec{x'_j}$ corresponding to vertex $i \in V(G'_j)$. By x_{i_j} we mean the element of $\vec{x}$ corresponding to vertex $i \in V(G_j)$. By $i_j = u(i_j = v)$ we simply mean that we want to assign the corresponding value of assumed eigenvector to $u, x_u(to\ v, x_v)$:

$$x_{i_j} = \begin{cases} x'_{i_j} & i \in V(G_j) \\ 0 & i_j = u \\ \displaystyle -\sum_{s \in \bigcup_{l=1}^{k} G_l,\ s \sim u_l (s \in V(G_l))} x_s & i_j = v \end{cases}$$

It is easy to verify that $\vec{x}$ is a Laplacian eigenvector for G. Such a construction is reversible,i.e. for a given G and $\vec{x}$ one can simply construct $\vec{x'}$ as shown below:

$$x_{i_j} = \begin{cases} x'_{i_j} & i \in V(G_j) \\ 0 & i_j = u \\ \displaystyle -\sum_{s \in \bigcup_{l=1}^{k} G_l,\ s \sim u_l (s \in G_l)} x_s & i_j = v \end{cases}$$

This conversion defines a bijection between the eigenspace of 1 in G and G' and thereforethe proof is complete. □

Corollary 4.*[6]*

Corollary 5. *Let G be a graph and u be a quasi-pendant vertex of G. If $v_1, v_2, ..., v_k$ be the vertices of degree one adjacent to u then:*

$$m_G(1) = m_{G\setminus\{v_2,...,v_k\}}(1) + k - 1$$

Corollary 6. *[10]Let G_1 be a graph on $n > 1$ vertices, and let G be the graph obtained from G_1 and $K_{1,s}$ $(s > 2)$ by joining a vertex u of G_1 to a vertex v of $K_{1,s}$. Then we have:*

(1) If v is a pendant vertex of $K_{1,s}$, then

$$m_G(1) = m_{G_1}(1) + s - 2;$$

(2) if v is the center of $K_{1,s}$, then

$$m_G(1) = m_{G_1 u:vw}(1) + s - 1.$$

Proof. This is a striaghtforward conclusion from lemmas 3 and 2.

□

Lemma 4. *If G is a graph, and u and v are two arbitrary adjacent vertices, and G' is a graph obtained from G by removing the edge $\{u, v\}$ and adding five new vertices, y, z, z', w, w' and connecting y to u and v, z to z' and to u and w to w' and v. Then we have:*

$$m_G(1) = m_{G'}(1)$$

Proof. Suppose $\vec{x}$ is an eigenvector of G corresponding to $\mu = 1$. We use equation 2 for u and v:

$$\begin{cases} \sum\limits_{i\sim v,\ i\neq u} x_i = (d_v - 1)x_v \\ \sum\limits_{i\sim y,\ i\neq v} x_i = (d_u - 1)x_u \end{cases}$$

Now define the function $f : \mathbb{R}^{\mathrm{n}} \mapsto \mathbb{R}^{\mathrm{n}+5}$, $f(\vec{x}) = \vec{x'}$ as follows:

$$x'_i = \begin{cases} x_i & i \in V(G) \\ x_v + x_u & i = y \\ 0 & i = z, w \\ -x_u & i = z' \\ -x_v & i = w' \end{cases}$$

It is easy to check that x' is an eigenvector for G' and also f is again an isomorphism between the eigenspace of 1 in two graphs and therefore the result follows.

$\square$

Lemma 5.*[11]Let G_1 and G_2 be two vertex-disjoint graphs and let $G = G_1u : vG_2$ be the graph obtained by joining the vertex u of the graph G_1 to the vertex v of the graph G_2 by an edge. Then:*

$$\phi(G) = \phi(G_1)\phi(G_2) - \phi(G_1)\phi(G_2\backslash\{v\}) - \phi(G_2)\phi(G_1\backslash\{u\}).$$

3 Multiplicity of 1 in Laplacian Spectrum of Trees

In this section we prove two main theorems which euqation holds in Faria's inequality. We introduce a method to easily compute the multiplicity of 1 in trees.

Theorem 1.

Suppose T is a tree. There is a normal tree T' with $O(T') < O(T)$(if $T \neq T'$) and subdivision tree T'' which have the properties:

$$m_T(1) = m_{T'}(1) + p(T) - q(T) = m_{T''}(1) + p(T) - q(T)$$

Proof. We prove the existence of T' by strong induction on the order of T and from that we conclude the existence of T''. If $O(T) = 1, 2$ or 3, $T = T'$ satisfy the result. Now suppose $O(T) \geq 4$. If T is a tree obtained from a tree by connecting one of its vertices to an endvertex of P_3, according to Lemma 2, removing this path from T won't change the multplicity of 1, however order of this new tree is strictly smaller than $O(T)$ and the result follows by induction hypothesis. So without loss of genereality we can assume that T does not contain any P_3 as previously mentioned. If T has a quasi-pendant vertex of degree $k > 2$, use Lemma 3 on one of the pendant neighbours of this vertex to obtain the graph $\tilde{T}$(as $\tilde{G}$ in Lemma 3). From the induction hypothesis we know that there are trees $T_1', T_2', ...T_{k-1}'$ which are normal and now we can consider a quasipendant vertex from each(of $T_i = P_2$ simplu ignore it) and use Lemma 3 to obtain T' as desired. The only case left is a tree which all of its quasi-pendanet vertices are of degree two. Now lemma 2 will give the desired T' by induction. It is easy to see that in each case the if T was not itself a normal tree, T' has an smaller order. To obtain T'' use lemma 4 on any two vertices of T' which are not of degree 2 and connected.

□

Theorem 2.

For any normal tree we have:

$$m_T(1) = p(T) - q(T).$$

Proof. First use lemma 3 to build a graph which has only one pendant vertex connected to each quasi pendant vertex to obtain the tree T'. Now we show that $m_{T'}(1) = 0$. We prove this by strong induction on order of T'. For $O(T') = 1, 2$ or 3 the result is obvious. So we can assume $O(T') \geq 4$. If T' has a quasi-pendant vertex of degree 2(say w), and its pendant neighbour is s and its other neighbour is t, do as follows: Use lemma 2 to add three new vertices, a, u and v which a is connected to w and u and u is connected to v and v is connected to t. Now consider G_1 is the component containing u when we remove the edge $\{u, v\}$ and G_2 the component containing v. According to 3 $P(G_1, 1) \neq 0$ and by induction hypothesis $P(G_2, 1) \neq 0$. Also using 3 $P(G_1 \backslash \{u\}, 1) = 0$. By lemma 5 we deduce that $P(T', 1) \neq 0$ and we gain the desired result. So any quasi-pendant vertex of T has degree greater than or equal to three. Now consider any quasi-pendant vertex of T' and use 3 to get the T_i''s. For each T_i' if the vertex other than the pendant one(u_1) has degree greater than two we can use the induction hypothesis. So it is sufficient to show that for any T_1' which $d_{u_1} = 2$, we have $m_{T_i'}(1) = 0$. Consider that the new vertex connected to u_1 is u_2. If $d_{u_2} > 3$ use 2 and by induction hyptothesis we have the result. Now because T was normal and T doesn't have any quasi-pendanet vertex with degree two, therefore $d_{u_2} = 3$. Suppose u_3 and v_1 are the other neighbours of u_2. Using 2 we have a new tree with $d_{u_2} = 2$. If $d_{u_3} \geq 3$ and $d_{v_1} \geq 3$ use 3 and the result follows according to induction hyptothesis. Also according to assumptions it is not the case that $d_{u_3} = d_{v_1} = 1$. If either of them has degree two, use 2 and the result follows by induction hyptothesis. So one of them should be of degree 1 and the other should be of degree greater than or equal to 3. Without loss of generality consider $d_{u_3} \geq 3$ and $d_{v_1} = 1$. But this is the same as the case for u and its pendant neighbour. Because this procedure cannot contunie infinitley the result follows.

□

Corollary 7.

For any tree T, one can compute the multiplicity of 1 using previous transformations.

Proof. According to theorem 1 and theorem 2 one can compute multiplicity of 1.

□

Remark 2. Because there is a similar equation for Signless Laplacian matrix as the equation 2, one can easily deduce that the multiplicity of 1 in Signless Laplacian spectrum is greater than or equal to the star degree of a graph.

In this section we give a lower bound for the multiplicity of 0 in adjacency spectrum of graphs. As we proved before there is a similar equation for graphs as equation 2:

$$\lambda x_v = \sum_{u \sim v} x_u$$

For this purpose we need two prove some lemmas.

Lemma 6. *Let G be a graph, u a quasi-pendant vertex of G and $v \in V(G)$ be a pendant vertex adjacent to u. Also suppose that $G \backslash \{u, v\}$ has k components which are $G_1, G_2, ..., G_k$. Now construct $G'_1, G'_2, ..., G'_k$ by adding u_i and v_i to them as u and v were connected in G and define $\widetilde{G} = \bigcup_{i=1}^{k} G'_i$, Then:*

$$\tilde{m}_G(0) = m_{\widetilde{G}}(0)$$

Proof. The proof is similar to the proof of lemma 3.

□

Lemma 7. *Suppose G_1 and G_2 are to graphs and G is the graph with $V(G) = V(G_1) \cup V(G_2) \cup \{w\}$, constructed by connecting arbitrary vertices $u \in G_1$ and $v \in G_2$ to a new vertex w. Then the graph G' which is obtained by merging u and v in G_1 and G_2 has the same multiplicity of 1 in Laplacian spectrum with G.*

Proof. The proof is similar to the proof of lemma 2.

□

Remark 3. Though it is not hard to prove that multplicity of 1 in Laplacian spectrum is greater than or equal to star degree of a graph, one can prove with using 6 which shows in a way star-degree is smaller than multiplicity of 1 in Laplacian spectrum and multiplicity of 0 in adjacency spectrum.

Remark 4.

Using lemmas 7 and 6 the multiplicity of 1 in trees is computable easily.

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